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Dynamic Model and Control of Mass-Distributed Tether Satellite System

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A set of partial differential equations is developed to describe the mass-distributed and extensible tether satellite system. Supplemented with a range-rate control algorithm and proper boundary conditions, the system can be studied either by the numerical integration of the equations or by considering the stationary motion (configuration) in the system. The former approach is not adequate because (if the integration is possible) it requires very high mathematical skill and huge amount of computations. On the contrary, the stationary configuration is easier to determine and provides more valuable information about the system as a whole. The stable stationary configuration is essentially all that is needed to carry out a tether mission in space. An iterative algorithm to compute the stationary configuration and a method of its stability analysis is proposed. Finally, a numerical simulation is given.

Nomenclature

A = tether-tube element
B = base satellite
$Bd\varphi\theta$ = spherical coordinate system
$Bxyz$ = orbital coordinate system
$\boldsymbol{C}$ = coefficient vector to be determined in the text
D = magnitude of $\boldsymbol{D}$, m
$\boldsymbol{D}$ = relative position vector, m
$\mathrm{d}t$, $\mathrm{d}s$ = differentials
E = elasticity modulus, kg-m/s^2
e = orbital eccentricity
$\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ = unit vectors of $Bd\varphi\theta$
$\boldsymbol{F}(F_1, F_2, F_3)$ = perturbing force vector induced acceleration on S, m/s^2
$\boldsymbol{f}(f_1, f_2, f_3)$ = perturbing force vector induced acceleration on A, m/s^2
$\boldsymbol{G}$ = differential operator
$\boldsymbol{g}$ = Earth's gravity acceleration vector, m/s^2
H = integration constant
$\boldsymbol{I}$ = unit matrix
k, k_1, k_2 = parameters in the control algorithm
l = deployed tether length, m
m = subsatellite's mass, kg
N = total number of iterations
$\boldsymbol{P}, \boldsymbol{Q}$ = system matrices
p = focal parameter, m
$\boldsymbol{R}$ = radius vector from the Earth center to A, m
$\boldsymbol{R}_B$ = radius vector from the Earth center to B, m
$\boldsymbol{R}_S$ = radius vector from the Earth center to S, m
S = subsatellite
s = arc length along the unstrained tether line, measured from B to A, m
T = tether tension, kg-m/s^2
t = time, s
$\boldsymbol{u}$ = local velocity of the tangential movement along the tether tube, m/s
$\boldsymbol{V}$ = velocity of A, m/s
V_r = radial component of the orbital velocity, m/s
V_u = tangential component of the orbital velocity, m/s
$\boldsymbol{W}$ = generalized perturbing term, m/s^2
$\boldsymbol{X}$ = state vector
$\boldsymbol{Y} = \{\boldsymbol{y}_{ij}\}$ = fundamental solution matrix
$\boldsymbol{y}$ = solution vector
$\boldsymbol{Z}$ = state vector
$\gamma (= \Delta l/\Delta l_0)$ = elongation where Δl is the strained length and Δl_0 is the unstrained length of the same tether element
θ = out-of-plane directional angle of $\boldsymbol{D}$, rad
λ = eigenvalue
μ = Earth's gravitational constant, m^3/s^2
ν = true anomaly of B, rad
ρ, ρ_0 = mass per unit length (linear density) of strained and unstrained tether, respectively, kg/m
φ = in-plane directional angle of $\boldsymbol{D}$, rad
ω = orbital angular velocity of circular orbit, rad/s
$\cdot$ = differentiation with respect to t
$'$ = differentiation with respect to s

Subscript

0 = variables in the stationary configuration

Introduction

AT the early stage of the tether satellite system (TSS) and still at the present time, the massless tether model has been adopted. What is studied by this model is the motion of S in relation to B, while the tether is assumed to be a straight line. Many valuable conclusions have been drawn from this model, such as the instability of the tether retrieval motion and a tension control law,[1] the syntheses of the length rate control algorithm,[2,3] the equilibrium and stationary states of the system,[4,5] the periodic motion in the elliptical orbit,[5] etc., just to mention a few. References 1–5 represent advances in TSS as a new space object.

However, the distributed mass over a real tether (especially a prolonged tether) exerts certain influence on the motion of the whole system. The tether is never a straight line in general, but has a curved configuration in space. The motion of each element on the tether line is equal in importance. The large number of the elements on the tether line or, in other words, the large system dimension inevitably complicates the system's dynamics. Therefore, a further study on TSS dynamics is desired. Beletsky and Levin[6] have done a comprehensive study on the dynamics of the mass-distributed tether system, mainly in the free or perturbed motion of the system in the Newtonian field. The difference between this paper and the works by Beletsky is that the author has developed a range-rate control algorithm (RRCA) so that the controlled motion is treated in this paper. The system is modeled by the partial differential equations with RRCA-type boundary conditions. The tether's elongation, tension, tangential motion, and spatial displacement are considered as

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the state variables in the system. The orbital and spherical coordinate systems, which have been used by the author in the past,[5,7] are continuously used in this paper to describe the massive tether system. As no analytical solution exists to these partial differential equations, the numerical algorithm and method of nonlinear dynamic system theory are adopted to the system analysis.

Mathematical Model of Mass-Distributed Tether System

In the mass-distributed tether system the state variables depend not only on time but also on the location of the point under consideration on the tether line. In other words, a mathematical model of such a system will consist of a set of partial differential equations. To derive the model, the following assumptions are made: the base satellite's orbit is Keplerian, and the uniform tether has no bending or torsional stiffness.

Consider a perfectly flexible tube, with which the tether line is coinciding in form at each time moment. The tether might have a tangential motion $\boldsymbol{u}$ with respect to the tube as if the tether is filling in the tube and flowing along it like a liquid. Therefore, the tube can imitate the form of the tether in space. The hypothetical tube is called the tether tube.

Consider an element A on the tube, which is localized by s and lengthened by $\mathrm{d}s$ (Fig. 1). The increase of the total mass of A equals to the difference between the mass flows through the end sections of A, that is,

$$\Delta(\rho\,\mathrm{d}s) = (\rho u\,\mathrm{d}t)_s - (\rho u\,\mathrm{d}t)_{s+\mathrm{d}s} \tag{1}$$

or in the differential expression

$$\dot{\rho} = -u\rho' - \rho u' \tag{2}$$

For $\rho = \rho_0\gamma^{-1}$ is by definition, and so $\dot{\rho} = -(\rho_0/\gamma^2)\dot{\gamma}$ and $\rho' = -(\rho_0/\gamma^2)\gamma'$; then Eq. (2) is transformed into

$$\dot{\gamma} = -u\gamma' + \gamma u' \tag{3}$$

Besides the tensions $\boldsymbol{T}(s)$ and $\boldsymbol{T}(s+\mathrm{d}s)$ exerted by the neighboring segments on the tube (Fig. 2), there are other forces on A, for example, the gravitational force $\rho\,\mathrm{d}s\cdot\boldsymbol{g}$ and the aerodynamic or electrodynamic perturbing force $\rho\,\mathrm{d}s\cdot\boldsymbol{f}$. $\boldsymbol{V} = \dot{\boldsymbol{R}}$; the motion of A is governed by

$$\frac{\mathrm{d}}{\mathrm{d}t}[\rho\,\mathrm{d}s(\boldsymbol{V}+\boldsymbol{u})] = \boldsymbol{T}(s+\mathrm{d}s) - \boldsymbol{T}(s) + \rho\,\mathrm{d}s(\boldsymbol{g}+\boldsymbol{f})$$

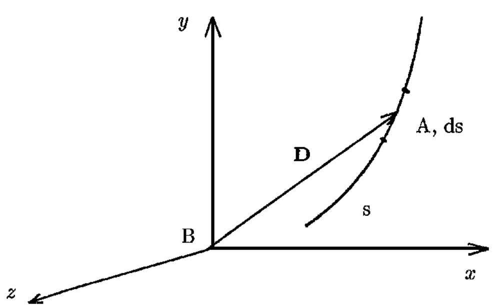

Fig. 1 Localization of tether element.

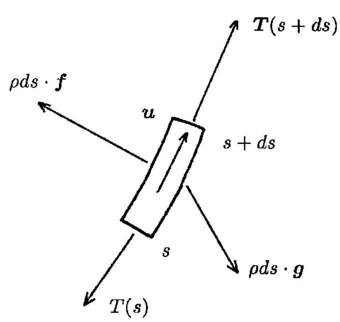

Fig. 2 Tether-tube element.

or in the expanded form

$$\rho\dot{\boldsymbol{V}} = \boldsymbol{T}' + \rho(\boldsymbol{g}+\boldsymbol{f}) - \rho\dot{\boldsymbol{u}} - \dot{\rho}\boldsymbol{u} - \dot{\rho}\boldsymbol{V} \tag{4}$$

substitute $g = -(\mu/R^3)\boldsymbol{R}$, $\boldsymbol{T} = (T/\gamma)\boldsymbol{D}'$, $\boldsymbol{u} = (u/\gamma)\boldsymbol{D}'$ into Eq. (4) to obtain

$$\ddot{\boldsymbol{R}} = (T/\rho_0)\boldsymbol{D}'' + \alpha\boldsymbol{D}' - (u/\gamma)\dot{\boldsymbol{D}}' + \boldsymbol{g} + \boldsymbol{f} + (\dot{\gamma}/\gamma)\dot{\boldsymbol{R}} \tag{5}$$

where

$$\alpha = (\gamma/\rho_0)(T/\gamma)' - \dot{u}/\gamma + 2u\dot{\gamma}/\gamma^2$$

In TSS, what is studied is the motion of the element A in relation to B, which is described by the relative position vector $\boldsymbol{D} = \boldsymbol{R} - \boldsymbol{R}_B$ so that

$$\ddot{\boldsymbol{D}} = \ddot{\boldsymbol{R}} - \ddot{\boldsymbol{R}}_B \tag{6}$$

Substitute Eq. (5) and $\ddot{\boldsymbol{R}}_B = -(\mu/R_B^3)\boldsymbol{R}_B$ into Eq. (6) to obtain

$$\ddot{\boldsymbol{D}} - \frac{T}{\rho_0}\boldsymbol{D}'' = \alpha\boldsymbol{D}' - \frac{u}{\gamma}\dot{\boldsymbol{D}}' + \boldsymbol{f} + \frac{\dot{\gamma}}{\gamma}\dot{\boldsymbol{R}} - \frac{\mu}{R^3}\boldsymbol{R} - \frac{\mu}{R_B^3}\boldsymbol{R}_B \tag{7}$$

The three components of Eq. (7) along $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ are the following:

$$\begin{aligned}
&\ddot{D} - D\dot{\theta}^2 - D(\dot{\nu}+\dot{\varphi})^2\cos^2\theta - \frac{T}{\rho_0}[D'' - D(\theta')^2 - D(\varphi')^2\cos^2\theta] \\
&= \frac{\mu}{R_B^3}D(3\cos^2\theta\sin^2\varphi - 1) + \alpha D' \\
&- \frac{u}{\gamma}[\dot{D}' - D\dot{\theta}\theta' - D(\dot{\nu}+\dot{\varphi})\varphi'\cos^2\theta] + f_1 + \frac{\dot{\gamma}}{\gamma}\dot{R}_1
\end{aligned} \tag{8}$$

$$\begin{aligned}
&D(\ddot{\nu}+\ddot{\varphi}) + 2\dot{D}(\dot{\nu}+\dot{\varphi}) - 2D(\dot{\nu}+\dot{\varphi})\dot{\theta}\tan\theta \\
&- \frac{T}{\rho_0}(D\varphi'' + 2D'\varphi' - 2D\varphi'\theta'\tan\theta) = 3\frac{\mu}{R_B^3}D\cos\varphi\sin\varphi \\
&+ \alpha D\varphi' - \frac{\mu}{\gamma}[\dot{D}\varphi' + D\dot{\varphi}' - D\varphi'\dot{\theta}\tan\theta + D'(\dot{\nu}+\dot{\varphi}) \\
&- D(\dot{\nu}+\dot{\varphi})\theta'\tan\theta] + \frac{1}{\cos\theta}\left(f_2 + \frac{\dot{\gamma}}{\gamma}\dot{R}_2\right)
\end{aligned} \tag{9}$$

$$\begin{aligned}
&D\ddot{\theta} + 2\dot{D}\dot{\theta} + D(\dot{\nu}+\dot{\varphi})^2\cos\theta\sin\theta \\
&- \frac{T}{\rho_0}[D\theta'' + 2D'\theta' + D(\varphi')^2\cos\theta\sin\theta] \\
&= -1.5\frac{\mu}{R_B^3}D\sin^2\varphi\sin 2\theta + \alpha D\theta' - \frac{u}{\gamma}[\dot{D}\theta' + D\dot{\theta}' \\
&+ 0.5D(\dot{\nu}+\dot{\varphi})\varphi'\sin 2\theta + D'\dot{\theta}] - f_3 - \frac{\dot{\gamma}}{\gamma}\dot{R}_3
\end{aligned} \tag{10}$$

where

$$\dot{R}_1 = \dot{D} + V_r\cos\theta\sin\varphi - V_u\cos\theta\cos\varphi$$

$$\dot{R}_2 = D(\dot{\nu}+\dot{\varphi})\cos\theta + V_r\cos\varphi + V_u\sin\varphi$$

$$\dot{R}_3 = -D\dot{\theta} + V_r\sin\theta\sin\varphi - V_u\sin\theta\cos\varphi$$

$$V_r = (\mu/p)^{0.5}e\sin\nu, \qquad V_u = (\mu/p)^{0.5}(1 + e\cos\nu)$$

The system of Eqs. (8–10) contains six unknown variables: $D(t,s)$, $\varphi(t,s)$, $\theta(t,s)$, $T(t,s)$, $\gamma(t,s)$, and $u(t,s)$. Three more equations are to be added as follows:

1) $\gamma = |\bar{D}'|$, or in explicit form,

$$\gamma^2 = (D')^2 + (D\varphi'\cos\theta)^2 + (D\theta')^2 \tag{11}$$

2) Hooke's law of elasticity

$$T = (\gamma - 1)E \tag{12}$$

3) Equation (3) of nondiscontinuity

$$u' = (1/\gamma)(\dot{\gamma} + u\gamma') \tag{13}$$

To integrate the system of Eqs. (8–13), a set of initial and boundary conditions has to be formed. The initial condition at $t = 0$ is determined by $D(0, s)$, $\varphi(0, s), \ldots, u(0, s)$, which have to satisfy the following equation:

$$D'\dot{D}' + D\varphi' \cos\theta(\dot{D}\varphi' \cos\theta + D\dot{\varphi}' \cos\theta - D\varphi'\dot{\theta} \sin\theta)$$
$$+ D\theta'(\dot{D}\theta' + D\dot{\theta}') = \gamma\dot{\gamma} \tag{14}$$

Furthermore, the boundary conditions are fixed to both ends of the tether line, at B ($s = 0$) and at S ($s = l$). For example, the speed u at $s = 0$ is specified by RRCA[5,7]:

$$u(t, 0) = \left.\frac{k\dot{\nu} + k_1\dot{\varphi} + k_2|\dot{\theta}|}{\dot{\nu} + \dot{\varphi}}\dot{\nu}D\right|_{s=l}$$
$$k \subset (-0.75, 0.75), \quad k_1 > 0, \quad k_2 > 0 \tag{15}$$

The boundary condition at $s = l$ is expressed through the equations of motion of S as follows:

$$\ddot{D} - D\dot{\theta}^2 - D(\dot{\nu} + \dot{\varphi})^2 \cos^2\theta$$
$$= \left(\mu/R_B^3\right)D(3\cos^2\theta \sin^2\varphi - 1) - (T/m)(D'/\gamma) + F_1 \tag{16}$$

$$D(\ddot{\nu} + \ddot{\varphi}) + 2\dot{D}(\dot{\nu} + \dot{\varphi}) - 2D(\dot{\nu} + \dot{\varphi})\dot{\theta}\tan\theta$$
$$= 1.5\left(\mu/R_B^3\right)d \sin 2\varphi - (T/m)(D\varphi'/\gamma) + F_2 \tag{17}$$

$$D\ddot{\theta} + 2\dot{D}\dot{\theta} + 0.5D(\dot{\nu} + \dot{\varphi})^2 \sin 2\theta$$
$$= -1.5\left(\mu/R_B^3\right)D \sin^2\varphi \sin 2\theta - (T/m)(D\theta'/\gamma) + F_3 \tag{18}$$

As soon as Eqs. (8–13) have been solved, the tether's configuration in *Bxyz* can be expressed by

$$x = D\cos\theta\cos\varphi, \qquad y = D\cos\theta\sin\varphi, \qquad z = -D\sin\theta$$

The mathematical model for inextensible tether system can be derived from Eqs. (8–13) by setting $\gamma = 1$, and a model for stationkeeping phase is obtained by setting $u = 0$. In particular, the mathematical model of the in-plane motion ($\theta = 0$) of inextensible tether in stationkeeping is given as

$$\ddot{D} - D(\dot{\nu} + \dot{\varphi})^2 - (T/\rho_0)[D'' - D(\varphi')^2]$$
$$= \left(\mu/R_B^3\right)D(3\sin^2\varphi - 1) + (T'/\rho_0)D' + f_1 \tag{19}$$

$$D(\ddot{\nu} + \ddot{\varphi}) + 2\dot{D}(\dot{\nu} + \dot{\varphi}) - (T/\rho_0)(D\varphi'' + 2D'\varphi')$$
$$= 3\left(\mu/R_B^3\right)D\cos\varphi\sin\varphi + (T'/\rho_0)D\varphi' + f_2 \tag{20}$$

$$(D')^2 + D^2(\varphi')^2 = 1 \tag{21}$$

There are two categories of the methods to analyze the mathematical model, namely, the quantitative approach and qualitative approach. The quantitative approach involves an integration of the differential equations in the model. As no analytical solution exists, the numerical integration has to be applied. However, the integration of partial differential equations, if available, requires very high mathematical skill and a lot of machine time. Besides, the integration along an individual trajectory cannot provide enough information about the system in the whole.

On the contrary, the qualitative approach applied to the stationary motion (configuration) deals with the ordinary differential equations and offers a general information for the whole picture of the system behavior. The qualitative approach is more suitable to the system and must be applied in the first place.

Stationary Configuration

Most tether missions in space are to be conducted within one or another stable stationary configuration of the system, for example, a configuration along or near the local vertical. Therefore, a rational question arises: is there a stationary configuration? If so, is the stationary configuration stable? The following sections are devoted to the solution to these questions.

Definition of Stationary Configuration

The stationary motion of mass-distributed tether system is defined as the motion in which all points of the system are rotating about the Earth along the concentric circles at the same orbital angular velocity. The tether forms a fixed configuration in *Bxyz*. The equations of the stationary configuration are obtained by substituting $\dot{\nu} = \omega = \text{const}$ and $\dot{D} = \ddot{D} = \dot{\varphi} = \ddot{\varphi} = 0$ into Eqs. (19–21) as follows:

$$D'' + (T'/T)D' - D(\varphi')^2 + 3(\rho_0/T)\omega^2 D\sin^2\varphi = -(\rho_0/T)f_1 \tag{22}$$

$$D\varphi'' + [2D' + (T'/T)D]\varphi' + 3(\rho_0/T)\omega^2 D\cos\varphi\sin\varphi$$
$$= -(\rho_0/T)f_2 \tag{23}$$

$$(D')^2 + D^2(\varphi')^2 = 1 \tag{24}$$

Equations (22–24) are to be integrated under a set of boundary conditions, for example,

$$D(0) = 0 \tag{25a}$$

$$D'(0) = 1 \tag{25b}$$

$$\varphi'(0) = 0 \tag{25c}$$

$$T(l)D'(l) = 3mD(l)\omega^2\sin^2\varphi(l) + F_1 \tag{25d}$$

$$T(l)D(l)\varphi'(l) = 3mD(l)\omega^2\sin\varphi(l)\cos\varphi(l) + F_2 \tag{25e}$$

Equations (25b) and (25c) assume that the initial part of the tether is a straight line.

Locally Vertical Stationary Configuration

The stationary configuration of an inextensible tether along the local vertical is characterized by $\varphi = 3\pi/2$, $l = D_S$, $\gamma = D' = 1$. Such a configuration implies that the perturbing force on the tether as well as on the subsatellite is also to be aligned with the local vertical. For simplicity, the perturbing force is assumed zero. In this case Eq. (23) is self-satisfied, and Eq. (22) becomes

$$T' + 3\rho_0\omega^2 D = 0$$

The integral of the preceding equation gives (note $\mathrm{d}s = \mathrm{d}D$)

$$T + \tfrac{3}{2}\rho_0\omega^2 D^2 = H \tag{26}$$

At $s = l$, $T = 3m\omega^2 l$ [see Eq. (25d)]. Equation (26) gives

$$H = 3m\omega^2 l + \tfrac{3}{2}\rho_0\omega^2 l^2$$

Then, $T(s)$ is computed by Eq. (26) for any point on the tether line. These data will be used as initial values to compute any (no locally vertical) stationary configuration under nonzero (f_1, f_2) and (F_1, F_2).

Computing Algorithm for Stationary Configuration of Inextensible Tether

Differentiating Eq. (24) with respect to s gives

$$D'D'' + DD'(\varphi')^2 + D^2\varphi'\varphi'' = 0 \tag{27}$$

Extract D'' and φ'' from Eqs. (22) and (23) and substitute into Eq. (27) to obtain

$$T' = -\rho_0\left[D'\left(3\omega^2 D\sin^2\varphi + f_1\right) + D\varphi'\left(1.5\omega^2 D\sin 2\varphi + f_2\right)\right] \tag{28}$$

In other words, the original system of Eqs. (22–24) now is replaced by a new system of Eqs. (22), (23), and (28). Introduce $\boldsymbol{X} = [x_1, x_2, x_3, x_4, x_5]^T$ with the components

$$x_1 = D, \qquad x_2 = D', \qquad x_3 = \varphi, \qquad x_4 = \varphi', \qquad x_5 = T$$

The new system can be transformed into the state-space notation as

$$\boldsymbol{X}' = \boldsymbol{G}(\boldsymbol{X}, \boldsymbol{W}) \tag{29}$$

where

$$W = \begin{cases} (f_1, f_2)^T & \text{for the tether line} \\ (F_1, F_2)^T & \text{for the subsatellite} \end{cases}$$

$$G_1 = x_2$$

$$G_2 = -\left(3\rho_0\omega^2 x_1 \sin^2 x_3 + G_5 x_2 + \rho_0 f_1\right)x_5^{-1} + x_1 x_4^2$$

$$G_3 = x_4$$

$$G_4 = -2x_2x_4x_1^{-1} - \left(G_5 x_4 + 1.5\rho_0\omega^2 \sin 2x_3 + \rho_0 x_1^{-1} f_2\right)x_5^{-1}$$

$$G_5 = -\rho_0\left[x_2\left(3\omega^2 x_1 \sin^2 x_3 + f_1\right) + x_1x_4\left(1.5\omega^2 x_1 \sin 2x_3 + f_1\right)\right]$$

To solve Eq. (29) of the stationary configuration, an iterative algorithm of N steps is applied.[5] Divide (f_1, f_2) and (F_1, F_2) by N to obtain

$$\Delta W = \begin{cases} (\Delta f_1 = f_1/N, \Delta f_2 = f_2/N)^T \\ (\Delta F_1 = F_1/N, \Delta F_2 = F_2/N)^T \end{cases}$$

Let $\boldsymbol{W}(0) = 0$ and $\boldsymbol{X}(0)$ be the solution of the locally vertical stationary configuration found in the preceding section. Suppose $\boldsymbol{X}(i)$ is a solution (or an approximation to the solution) and $\boldsymbol{W}(i)$ is the perturbing term in ith step in the iteration process, respectively. Let $\boldsymbol{X}(i+1) = \boldsymbol{X}(i) + \Delta\boldsymbol{X}(i+1)$ and $\boldsymbol{W}(i+1) = \boldsymbol{W}(i) + \Delta\boldsymbol{W}$ be the solution and perturbing term in $i+1$ step. Substitute $\boldsymbol{X}(i+1)$ and $\boldsymbol{W}(i+1)$ into Eq. (29):

$$\boldsymbol{X}'(i) + \Delta\boldsymbol{X}'(i+1) = \boldsymbol{G}[\boldsymbol{X}(i) + \Delta\boldsymbol{X}(i+1), \boldsymbol{W}(i) + \Delta\boldsymbol{W}]$$

Expand the preceding expression, and retain only the first-order terms of the variations

$$\Delta\boldsymbol{X}'(i+1) = \left(\frac{\partial\boldsymbol{G}}{\partial\boldsymbol{X}}\right)_i \Delta\boldsymbol{X}(i+1) + \left(\frac{\partial\boldsymbol{G}}{\partial\boldsymbol{W}}\right)_i \Delta\boldsymbol{W} + \boldsymbol{r}(i)$$

$$\boldsymbol{r}(i) = \boldsymbol{G}[\boldsymbol{X}(i), \boldsymbol{W}(i)] - \boldsymbol{X}'(i) \tag{30}$$

Equation (30) is a nonhomogeneous equation. Its general solution is expressed in the form as

$$\Delta\boldsymbol{X}(i+1) = \boldsymbol{y}(i+1) + \boldsymbol{Y}(i+1)\boldsymbol{C} \tag{31}$$

where vector $\boldsymbol{C}$ is to be determined later. Other terms in Eq. (31) are determined as follows:

$$\boldsymbol{y}'(i+1) = \left(\frac{\partial\boldsymbol{G}}{\partial\boldsymbol{X}}\right)_i \boldsymbol{y}(i+1) + \left(\frac{\partial\boldsymbol{G}}{\partial\boldsymbol{W}}\right)_i \Delta\boldsymbol{W} + \boldsymbol{r}(i)$$

$$\boldsymbol{y}(i+1)|_{s=0} = \boldsymbol{0} \tag{32a}$$

$$\boldsymbol{Y}'(i+1) = \left(\frac{\partial\boldsymbol{G}}{\partial\boldsymbol{X}}\right)_i \boldsymbol{Y}(i+1), \qquad \boldsymbol{Y}(i+1)|_{s=0} = \boldsymbol{I} \tag{32b}$$

The solution $\boldsymbol{X}(i+1) = \boldsymbol{X}(i) + \Delta\boldsymbol{X}(i+1)$ is $\boldsymbol{C}$ dependent [see Eq. (31)]. Therefore, $\boldsymbol{C}$ can be determined by the following boundary conditions that the solution has to satisfy

$$D(i+1)|_{s=0} = 0 \tag{33a}$$

$$D'(i+1)|_{s=0} = 1 \tag{33b}$$

$$\varphi'(i+1)|_{s=l} = 0 \tag{33c}$$

$$D'(i+1)|_{s=l} = m\left.\frac{3\omega^2 D(i+1)\sin 2\varphi(i+1) + F_1(i+1)}{|T(i+1)|}\right|_{s=l} \tag{33d}$$

$$D'(i+1)\varphi'(i+1)|_{s=l} = m\left.\frac{1.5\omega^2 D(i+1)\sin 2\varphi(i+1) + F_2(i+1)}{|T(i+1)|}\right|_{s=l} \tag{33e}$$

When $\boldsymbol{C}$ is obtained, the solution-increment $\Delta\boldsymbol{X}(i+1)$ in Eq. (31) can be recalculated, and the solution $\boldsymbol{X}(i+1)$ is obtained. Performing N steps in the iteration process, the stationary solution under given (f_1, f_2) and (F_1, F_2) is computed. If N is sufficiently large, the solution's accuracy is expected to be very high.

Stability Analysis

A computed stationary configuration is realizable if it is stable. The concept of variation equation is used to investigate the stability of the stationary configuration. Suppose D_0, φ_0, T_0 are the variables in the stationary configuration. ΔD, $\Delta\varphi$, ΔT are the small deviations of the state variables from the stationary configuration. Substitute $D = D_0 + \Delta D$, $\varphi = \varphi_0 + \Delta\varphi$, and $T = T_0 + \Delta T$ into Eqs. (19–21), and expand the obtained expressions about the stationary configuration. In the expansion only the linear terms of ΔD, $\Delta\varphi$, ΔT and their derivatives are retained so that the variation equations are obtained as follows:

$$\Delta\ddot{D} - 2D_0\omega\Delta\dot{\varphi} - \frac{T_0}{\rho_0}\Delta D'' - \frac{T_0'}{\rho_0}\Delta D' + \frac{2T_0D_0\varphi_0'}{\rho_0}\Delta\varphi' - \frac{D_0'}{\rho_0}\Delta T' - \left[3\omega^2\sin^2\varphi_0 - \frac{T_0(\varphi_0')^2}{\rho_0}\right]\Delta D - 3\omega^2 D_0\sin 2\varphi_0\cdot\Delta\varphi - \frac{D_0'' - D_0(\varphi_0')^2}{\rho_0}\Delta T = 0 \tag{34}$$

$$D_0\Delta\ddot{\varphi} + 2\omega\Delta\dot{D} - \frac{T_0D_0}{\rho_0}\Delta\varphi'' - \frac{2T_0D_0' + T_0'D_0}{\rho_0}\Delta\varphi' - \frac{2T_0\varphi_0'}{\rho_0}\Delta D' - \frac{D_0\varphi_0'}{\rho_0}\Delta T' - \frac{3}{2}\omega^2\sin 2\varphi_0\Delta D - \frac{T_0\varphi_0'' + T_0'\varphi_0'}{\rho_0}\Delta D - 3\omega^2 D_0\cos 2\varphi_0\Delta\varphi - \frac{D_0\varphi_0'' + 2D_0'\varphi_0'}{\rho_0}\Delta T = 0 \tag{35}$$

$$D_0'\Delta D' + D_0^2\varphi_0'\Delta\varphi' + D_0(\varphi_0')^2\Delta D = 0 \tag{36}$$

Assume the solutions of Eqs. (34–36) have the following form:

$$\Delta\varphi(s,t) = \Delta\varphi(s)e^{\lambda t}, \qquad \Delta D(s,t) = \Delta D(s)e^{\lambda t}$$

$$\Delta T(s,t) = \Delta T(s)e^{\lambda t}$$

Substitute the preceding expressions into the system of Eqs. (34–36) to obtain

$$T_0\Delta D''(s) + T_0'\Delta D'(s) + \left[3\omega^2\rho_0\sin^2\varphi_0 - T_0(\varphi_0)^2 - \rho_0\lambda^2\right]\Delta D(s) - 2T_0D_0\varphi_0'\Delta\varphi'(s) + D_0'\Delta T'(s) + \rho_0\left(3\omega^2 D_0\sin 2\varphi_0 + 2D_0\omega\lambda\right)\Delta\varphi(s) + \left[D_0'' - D_0\left(\varphi_0' - D_0(\varphi_0')^2\right)\right]\Delta T(s) = 0 \tag{37}$$

$$2T_0\varphi_0'\Delta D'(s) + \left(T_0\varphi_0'' + T_0'\varphi_0' + 1.5\omega^2\rho_0\sin 2\varphi_0 - 2\omega\rho_0\lambda\right)\Delta D(s) + T_0D_0\Delta\varphi''(s) + (2T_0D_0' + T_0'D_0)\Delta\varphi(s) + \left(3\omega^2 D_0\rho_0\cos 2\varphi_0 - D_0\rho_0\lambda^2\right)\Delta\varphi(s) + D_0\varphi_0'\Delta T'(s) + (D_0\varphi_0'' + 2D_0'\varphi_0')\Delta T(s) = 0 \tag{38}$$

$$D_0'\Delta D'(s) + D_0(\varphi_0')^2\Delta D(s) + D_0^2\varphi_0'\Delta\varphi'(s) = 0 \tag{39}$$

Introduce $Z_1 = \Delta D(s, \lambda)$, $Z_2 = \Delta D'(s, \lambda)$, $Z_3 = \Delta\varphi(s, \lambda)$, $Z_4 = \Delta\varphi'$, and $Z_5 = \Delta T(s, \lambda)$ so that the system of Eqs. (37–39) is rewritten as

$$\boldsymbol{PZ}' = \boldsymbol{QZ} \tag{40}$$

where

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ T_0'' & T_0 & -2T_0 D_0 \varphi_0' & 0 & D_0' \\ 2T_0\varphi_0' & 0 & 2T_0 D_0' + T' D_0 & T_0 D_0 & D_0 \varphi_0' \\ D_0' & 0 & D_0^2 \varphi_0' & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{Q} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \rho_0\lambda^2 + T_0(\varphi_0')^2 - 3\rho_0\varpi^2\sin^2\varphi_0 & 0 & -2D_0\omega\rho_0\lambda - 3\omega^2 \times D_0\rho_0\sin 2\varphi_0 & 0 & D_0(\varphi_0')^2 - D_0'' \\ 2\rho_0\omega\lambda - T_0\varphi'' - T_0'\varphi_0' - 1.5\omega^2\rho_0\sin 2\varphi_0 & 0 & D_0\rho_0\lambda^2 - 3\omega^2 D_0 \times \rho_0\cos 2\varphi_0 & 0 & -D_0\varphi_0'' - 2D_0'\varphi_0' \\ -D_0(\varphi_0')^2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The boundary conditions for $\boldsymbol{Z}(s, \lambda)$ are obtained by linearization of the original boundary conditions (33a–33e) as follows:

$$Z_1(0, \lambda) = 0 \tag{41a}$$

$$Z_2(0, \lambda) = 0 \tag{41b}$$

$$Z_4(0, \lambda) = 0 \tag{41c}$$

$$3m\omega^2\sin^2\varphi_0 Z_1(l, \lambda) + 3m\omega^2 D_0\sin 2\varphi_0 Z_3(l, \lambda) - T_0 Z_2(l, \lambda) - D_0 Z_5(l, \lambda) = 0 \tag{41d}$$

$$(1.5m\omega^2\sin 2\varphi_0 - T_0\varphi_0')Z_1(l, \lambda) + 3m\omega^2 D_0\cos 2\varphi_0 \cdot Z_3(l, \lambda) - D_0 T_0 Z_4(l, \lambda) - D_0\varphi_0' Z_5(l, \lambda) = 0 \tag{41e}$$

The fundamental solution-matrix $\boldsymbol{Y}$ of Eq. (40) is defined as

$$\boldsymbol{PY}' = \boldsymbol{QY}, \qquad \boldsymbol{Y}(0) = \boldsymbol{I} \tag{42}$$

Then, the general solution of Eq. (40) can be expressed by $\boldsymbol{Z} = \boldsymbol{YC}$ or in expanded form:

$$Z_i(s, \lambda) = \sum_{j=1}^{5} y_{ij}(s, \lambda) \cdot C_j, \qquad i = 1, 2, \ldots, 5 \tag{43}$$

Substitute $\boldsymbol{Z}(s, \lambda)$ into Eqs. (41a–41e) to obtain

$$\sum_{j=1}^{5} y_{j_1}(0, \lambda)C_j = 0 \tag{44a}$$

$$\sum_{j=1}^{5} y_{j_2}(0, \lambda)C_j = 0 \tag{44b}$$

$$\sum_{j=1}^{5} y_{j_4}(0, \lambda)C_j = 0 \tag{44c}$$

$$3m\omega^2\sin^2\varphi_0 \sum_{j=1}^{5} y_{j_1}(l, \lambda)C_j + 3m\omega^2 D_0\sin 2\varphi_0 \sum_{j=1}^{5} y_{j_3}(l, \lambda)C_J - T_0 \sum_{j=1}^{5} y_{j_2}(l, \lambda)C_J - D_0 \sum_{j=1}^{5} y_{j_5}(l, \lambda)C_J = 0 \tag{44d}$$

$$\left(1.5m\omega^2\sin 2\varphi_0 - T_0\varphi_0'\right) \sum_{j=1}^{5} y_{j_1}(l, \lambda)C_j + 3m\omega^2 D_0\cos 2\varphi_0 \sum_{j=1}^{5} y_{j_3}(l, \lambda)C_J - D_0 T_0 \sum_{j=1}^{5} y_{j_4}(l, \lambda)C_J - D_0\varphi_0' \sum_{j=1}^{5} y_{j_5}(l, \lambda)C_J = 0 \tag{44e}$$

The system of Eqs. (44a–44e) is linear with respect to C_j, $j = 1, 2, \ldots, 5$. For the nontrivial solution of C_j, the determinant of the system must be zero, that is,

$$\triangle(\boldsymbol{Y}, \lambda) = 0 \tag{45}$$

The value λ, satisfying Eq. (45), is the eigenvalue of the stationary configuration. To obtain all eigenvalues, Eqs. (42–45) are to be solved jointly and repeatedly. If the real parts of all eigenvalues are negative, the stationary configuration is stable.

Quasi-Stationary Motion in Tether Deployment and Retrieval Phases

During the deployment/retrieval phase, the tether reeling speed is varying; therefore, the length of the tether as well as the boundary conditions is also varying. Therefore, it is impossible to have any stationary motion at all. However, if the speed is small a quasi-stationary configuration can be suggested, which is obtained by setting $\dot{D} = \ddot{D} = 0$, $\dot{\varphi} = \ddot{\varphi} = 0$ in Eqs. (8) and (9) as follows (also assume $\theta = 0$ and $f_1 = f_2 = 0$ for simplicity):

$$D'' - D(\varphi')^2 + \frac{T'}{T}D' + 3\rho_0\omega^2\frac{D\sin^2\varphi}{T} - \rho_0\frac{D'}{T}\dot{u} + \rho_0\omega\frac{D\varphi'}{T}u = 0 \tag{46}$$

$$D\varphi'' + 2D'\varphi' + \frac{T'}{T}D\varphi' + 1.5\rho_0\omega^2\frac{D\sin 2\varphi}{T} - \rho_0\frac{D\varphi'}{T}\dot{u} - \rho_0\omega\frac{D'}{T}u = 0 \tag{47}$$

The equation for T' is obtained similarly as for Eq. (28):

$$T' = -\rho_0(3\omega^2 DD'\sin^2\varphi + 1.5\omega^2 D^2\varphi'\sin 2\varphi - \dot{u}) \tag{48}$$

The quasi-stationary configuration of Eqs. (46–48) and its stability can be investigated by the similar algorithms described in the preceding sections.

Numerical Simulation

The model and the iterative algorithm developed in the preceding sections have been numerically simulated to compute the stationary configuration of a mass-distributed tether system. In the simulation the base satellite is assumed to be on a circular 300-km-altitude orbit. The diameter of the tether is 1 mm. The linear mass density of the steel-made tether is 0.00632 kg/m, and for a Kevlar®-made tether the linear density is 0.00116 kg/m. The subsatellite's mass is 500 kg. The total number of iterations in the computation is 500. In the stationkeeping only the aerodynamic perturbing forces on the tether as well as on the subsatellite are taken into account, and the air drag model is the conventional velocity-squared law. In the deployment and retrieval only the tether reeling speed and acceleration are considered, which are designed by RRCA to be $k\omega l$ and $k^2\omega^2 l$, respectively. The fourth-order Runge–Kutta method has been adopted to integrate the variation equations in the iterative process.

Figure 3 shows the stationary configuration in stationkeeping. There are five curves corresponding to the five tether's lengths of 20, 40, 60, 80, and 100 km, respectively. As the tether's length increases, the air density at the subsatellite is growing rapidly leading to far greater deviation of the tether line from the local vertical. Figures 4 and 5 show the quasi-stationary configurations in the deployment ($k > 0$) and retrieval ($k < 0$). For each value of k, there are five curves corresponding to five tether's lengths of 20, 40, 60, 80, and 100 km. All of the tether configurations in the deployment are situated on the left-bottom part on the orbital plane, and all tether configurations in the retrieval are on the right-bottom part. As parameter k increases, that is the tether deployment or retrieval speed increases, the tether's configurations deviate away from the local vertical as a result of increasing of the Coriolis force. All of these facts are similar to those observed in the massless tether model.[5]

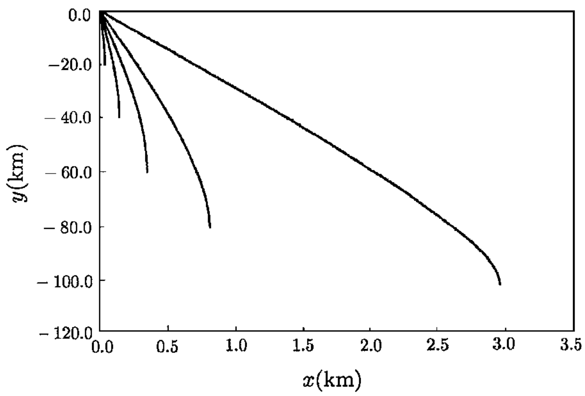

Fig. 3 Stationary configuration in stationkeeping.

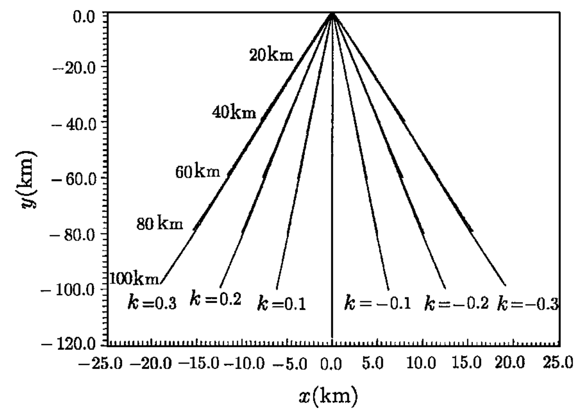

Fig. 4 Quasi-stationary configuration of steel-made tether.

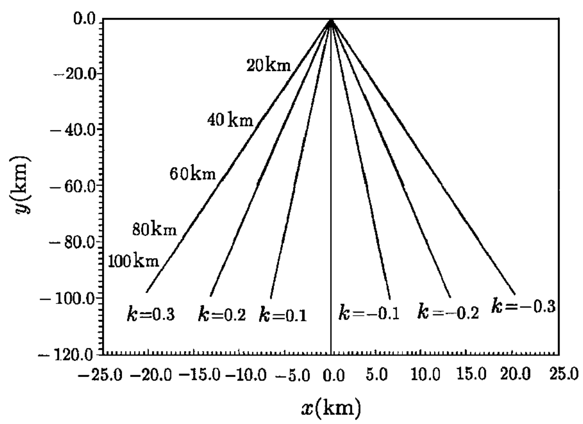

Fig. 5 Quasi-stationary configuration of Kevlar-made tether.

Conclusions

A mass-distributed tether is described by a system of partial differential equations complemented by a control law and boundary conditions. These equations will have a clearer form if they are written in a spherical coordinate system, and the concept of the relative position vector is used. However, the integration of these equations either by analytical solution or by numerical method presents enormous difficulty.

On the other hand, the stationary configuration of the system is easier to deal with. An iterative algorithm is applied to compute the stationary configuration, which is a boundary-value problem. The stability of the stationary configuration is studied by considering the variation equation, which is again a kind of two-point boundary-value problem to determine the eigenvalue of the stationary configuration. If the real parts of all eigevalues are negative, the stationary configuration is stable. Only the stable stationary configuration can be a working base for the tether mission in space. A numerical simulation demonstrates the potential of the mathematical model and the stationary configuration of the mass-distributed tether system developed in this paper.

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